deflections of the inner surface of the ring will coincide with the asymptotic behavior of the functions $H_{0n}(\tau)$ in cases b) and c). Consequently in the collapse of a ring of ideal incompressible liquid perturbations of general form develop similarly to irrotational perturbations.

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APPLICATION OF EXACT SOLUTIONS OF THE "SHALLOW WATER" EQUATIONS TO THE EXPLANATION OF THE SIMPLEST FLOWS

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Stationary solutions of the differential equations of the theory of "shallow water" with axial symmetry are given by the implications of these equations:

$$rhu = Q = Q_0/2\pi\rho = \text{const}; \tag{1}$$

$$rv = D = \text{const}; \tag{2}$$

$$\frac{u^2+v^2}{2} + gh = C = \text{const}, \tag{3}$$

where $h=h(r)$ is the height of an incompressible fluid layer of density ρ, $u=u(r)$, $v=v(r)$ are, respectively, the radial and circumferential components of the fluid velocity vector which is considered constant along the vertical in the whole layer $0 \le z \le h(r)$ in the "shallow water" approximation, g is the acceleration of gravity, and Q_0 is the fluid discharge through any section $r = \text{const} \ge r_0$.

From (1)-(3) we have the relationship

$$\frac{Q^2}{2r^2} = \frac{h^2(C-gh)}{1+\varkappa^2 h^2} = \varphi(h, C, \varkappa),\ \varkappa = D/Q, \tag{4}$$

which implicitly defines the dependence $h=h(r)$ for known Q, C, D.

Graphs of the function $\varphi(h, C, \varkappa)$ are presented in Fig. 1 for $\varkappa = 0$ and $\varkappa = 1$.

The method of the graphical determination of the dependence $h=h(r)$ is evident from (4), and it is also clear from Fig. 1 that a stationary axisymmetric solution of the "shallow water" equations exists only for

$$r > r_*(Q, C, \varkappa) > r_*(Q, C, 0) = (\sqrt{27/8})gQC^{-3/2}.$$

Hence (4) yields two solutions corresponding to two different fluid flow regimes.

The first flow regime corresponds to the dependence $h=h(r)$, determined from (4) for $0 < h < h_*(C, \varkappa)$, and the second for $h_*(C, \varkappa) < h < C/g$. Here $h_*(C, \varkappa)$ denotes the value of h for which $\varphi(h, C, \varkappa)$ reaches the maximum. It can be seen that for $\varkappa \ne 0$

$$h_*(C, \varkappa) < h(C, 0) = 2C/3g.$$

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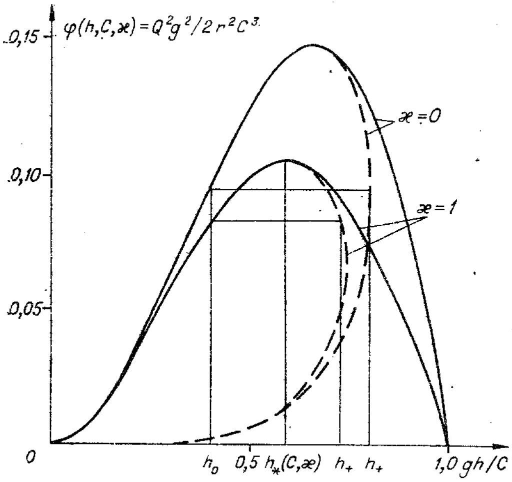

Fig. 1

The first regime is supercritical since the condition $|u| > c = \sqrt{gh}$ is satisfied, while the second regime in subcritical.

In the first flow regime we have an asymptotic as $r \to \infty$

$$h(r) \simeq \frac{|Q|}{\sqrt{2C}} \frac{1}{r}, \quad u(r) \simeq \operatorname{sgn} Q \cdot \sqrt{2C}. \tag{5}$$

On the other hand, in the second regime the fluid level h(r) is raised with the growth of r and as $r \to \infty$

$$h(r) \to C/g, \quad u(r) \simeq Qg/Cr. \tag{6}$$

The solution is determined uniquely from (1)-(4) by giving h(r), u(r), v(r) at any point $r = r_0$ since these conditions govern whether $h = h(r)$ belongs to one of the two branches of the curve (4).

Solutions of the system of conservation laws underlying the theory of "shallow water" can have a discontinuity, a "water jump." They are on the lines $r = \text{const}$ in the case of stationary axisymmetric flows.

The magnitude of the fluid discharge Q_0 and the tangential momentum component DQ_0/r are evidently continuous on the discontinuity. This means that the relations (1) and (2) are satisfied even in the discontinuous solutions and, in particular, that the quantity $\varkappa$ is constant in the whole flow.

The third condition which must be posed on the discontinuity is not quite so evident and can, in principle, depend on the conditions of the problem. The crux of the "shallow water" approximation, which replaces the momentum distribution along the vertical by its mean value and neglects the momentum loss because of fluid friction on the bottom, would be taken into account in solving this problem. In substance, a jump in the fluid level for a horizontal bottom means that at this site the friction on the bottom initiates the occurrence of a stream transformation zone in which the internal fluid friction retards the upper fluid layers, hence producing a rise in the fluid level.

The assumption about conservation of the total momentum of the fluid layer at the discontinuity can be considered most natural and comparatively exact within the framework of the "shallow water" theory. Hence, we shall require continuity of the radial component of the momentum flux tensor at the discontinuity, i.e., continuity of the quantity

$$j = \rho\left(\frac{g}{2} h^2 + hu^2\right).$$

Let us note that continuity of the quantity j at a jump contradicts the condition (3) and, hence, the quantity C undergoes a discontinuity at the jump. Let $h_0 < h_*(C, \varkappa)$ and h_+ be values of the fluid level height on different sides of the discontinuity on a circle of radius r; here h_0 is upstream.

Then the continuity condition for the momentum flux results in the equation

$$\frac{g}{2} h_+^2 + \frac{2\varphi(h_0, C, \varkappa)}{h_+} = \frac{g}{2} h_0^2 + \frac{2\varphi(h_0, C, \varkappa)}{h_0},$$

which is easily solved by taking account of (1) and (2). We obtain the formulas

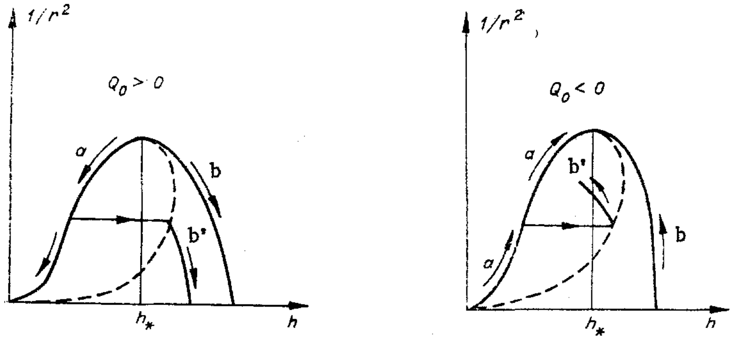

Fig. 2

$$\frac{h_+}{h_0} = K = \frac{\sqrt{1+8\alpha}-1}{2}, \quad \alpha = \frac{2\varphi(h_0, C, \varkappa)}{h_0^3 g} = \frac{2\left(\frac{C}{gh}-1\right)}{1+\varkappa^2 h_0^2}.$$

A stable jump is just the passage from the supercritical zone (before the jump) to the subcritical zone (after the jump) in the flow direction. It is hence necessary to assume that $h_0 < h_*(C, \varkappa)$, which results in the conditions $\alpha > 1$, $K > 1$, $h_+ > h_0$. Therefore, a value of the fluid level $h_+ = h_+(r)$ corresponds to each $h_0 = h(r) < h_*(C, \varkappa)$, and the set of points $\{h_+(r), Q^2g^2/2C^3r^2\}$ forms a curve of shock passage from the supercritical to the subcritical flow zones. The dashed lines in Fig. 1 show two such curves corresponding to the cases $\varkappa = 0$ and $\varkappa = 1$.

Therefore, the problem of the fluid spreading over the horizontal plane with given $h(r_0)$, $u(r_0)$, $v(r_0)$ has an uncountable set of solutions within the framework of the "shallow water" theory; one is continuous and the uncountable set discontinuous (the discontinuity can be set at any point $r > r_0$) for $h(r_0) < h_*(C, \varkappa)$.

The solution is unique in the case $h(r_0) > h_*(C, \varkappa)$. Possible kinds of solutions (and flows) are shown schematically in Fig. 2; the arrow indicates the displacement of points during downstream motion.

Therefore, the "shallow water" theory cannot determine the flow pattern uniquely for $h(r_0) < h_*(C, \varkappa)$ and additional reasoning is required to refine it.

To this end, let us consider the conditions for applicability of the "shallow water" approximation in our problem by considering the fluid to possess low viscosity.

For the "shallow water" approximation to be applicable it is necessary that the total head $gh^2/2 + hu^2$ of the fluid layer substantially exceed the viscous stress which we estimate as $|\nu h\partial u/\partial z| \sim \nu|u|$, where ν is the coefficient of kinematic viscosity.

Thus we have the applicability condition

$$gh^2/2 + hu^2 \gg \nu|u|,$$

or

$$\frac{h|u|}{\nu} + \frac{h|u|}{\nu}\frac{gh^2}{u^2} = \mathrm{Re}\left(1+\frac{1}{\mathrm{Fr}}\right) \gg 1,$$

where $\mathrm{Re} = h|u|/\nu = |Q|/r\nu$ is the Reynolds number; $\mathrm{Fr} = u^2/gh$ is the Froude number.

Taking account of the asymptotic behavior of the solutions (5) and (6) as $r \to \infty$, we conclude that

$$\mathrm{Re} \to 0,\ \mathrm{Fr} \to \infty,\ E = \mathrm{Re}\,(1 + 1/\mathrm{Fr}) \to 0$$

in the case of the supercritical flow ($h(r_0) < h_*(C, \varkappa)$) and

$$\mathrm{Re} \to 0,\ \mathrm{Fr} \to 0,\ E = \mathrm{Re}\,(1 + 1/\mathrm{Fr}) \to \infty \quad \text{as} \quad r \to \infty$$

in the case of a subcritical flow ($h(r_0) > h_*(C, \varkappa)$). Hence, it is clear that the conditions for applicability of this approximation as r grows are improved for the subcritical flow and degraded for the supercritical flow. The supercritical flow is therefore not realized for all $r > r_0$ and it can be assumed that the flow goes from the supercritical to the subcritical regime by a jump at namely that point r at which the conditions of applicability of the "shallow water" theory are spoiled, i.e., where the effect of viscosity is substantial and the quantity E becomes sufficiently small:

$$E = \mathrm{Re}\,(1 + 1/\mathrm{Fr}) = E_*. \tag{7}$$

Within the framework of such a model, the quantity E_* should be universal and independent of the discharge Q_0, the initial conditions, and the specific fluid. The quantity E_* should be determined from test or from a more general model. The condition (7) for a known E_* determines $h_0 < h_*(C, \varkappa)$ and the coordinate r of the jump where the change in flow regimes occurs.

This simple model explains the qualitative and quantitative characteristics of the two simplest flows fairly well: the spreading of a fluid supplied by a source with a constant discharge $Q_0 > 0$ over a horizontal plane, and the overflow of a fluid lying on an infinite horizontal plane, through a circular hole.

The first problem is illustrated by the customary spreading of water over the horizontal bottom of a basin during the fall of a water jet from a faucet on it; a more accurate illustration is obtained upon replacement of the basin by a horizontal flat smooth glass, metal, plexiglass, etc. surface of comparatively small size (~30 × 30 cm). The second problem is associated with the final stage in the process of water overflowing from a bath through a circular overflow hole.

Keeping the first problem in mind, it is natural to set $\varkappa = 0$. It is seen that the two regimes indicated in Fig. 2 are possible for $Q_0 > 0$; the regime b and the regime $a \to b'$ with the water jump at the point r determined from condition (7). Both regimes are realized in practice; however, the second possibility, with the formation of a jump, is accomplished for a jet falling from even a low height.

In this case $Fr > 1$ (since $h(r_0) < h_*$) and $Fr \to \infty$ as $r \to \infty$. The water jump occurs for sufficiently small h_0 and large r so that the quantity $1/Fr$ in (7) can apparently be neglected. We then have from (7)

$$\mathrm{Re} = \mathrm{Re}_* = Q_0/2\pi\rho\nu r = Q_0/2\pi\eta r. \tag{8}$$

The simplest measurements under domestic conditions result in the conclusion that $Re_* = E_* \simeq 150$. If this number is taken as Re_*, then (8) is satisfied quite well as the different problem parameters are varied, i.e., the discharge Q_0, the height of jet fall (the quantity C).

Within the framework of the "shallow water" theory, the stability conditions admitted two regimes also in the overflow problem: b and $a \to b'$ (see Fig. 2, $Q_0 < 0$). However, the last regime should be acknowledged as unrealistic since the "shallow water" theory is used in the regime a as $r \to \infty$, where $E \to 0$. Therefore, for $Q_0 < 0$ one single overflow regime remains b. For a given height h of the fluid layer at infinity, the fluid discharge $Q_0 < 0$ through any section $r = r_0$ is bounded by $|Q_0| < r_0 h \sqrt{\frac{8}{27} gh}$. For $h \to 0$ the maximum discharge through a hole of radius r_0 tends to zero and becomes less than the free discharge of the overflow hole. In this case a breach in the water surface occurs at the middle of the overflow hole.

As regards the fluid twisting which hence ordinarily occurs, it is explained, in our opinion, by the nonsymmetry of the real conditions of the problem as well as by the diminution in the discharge through the hole as the fluid twists. This problem has been studied both theoretically and experimentally. A quite complex flow model has been constructed in [1], which nevertheless does not yield good agreement with the results of experiments [2].

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